

These distributions are shown in Fig. 2.5-1. If both viscosities are the same, then the velocity distribution is parabolic, as one would expect for a pure fluid flowing between parallel plates (see Eq. 2B.3-2).

The *average velocity* in each layer can be obtained and the results are

$$\langle v_z^I \rangle = \frac{1}{b} \int_{-b}^0 v_z^I dx = \frac{(p_0 - p_L)b^2}{12\mu^I L} \left(\frac{7\mu^I + \mu^{II}}{\mu^I + \mu^{II}} \right) \quad (2.5-20)$$

$$\langle v_z^{II} \rangle = \frac{1}{b} \int_0^b v_z^{II} dx = \frac{(p_0 - p_L)b^2}{12\mu^{II} L} \left(\frac{\mu^I + 7\mu^{II}}{\mu^I + \mu^{II}} \right) \quad (2.5-21)$$

From the velocity and momentum-flux distributions given above, one can also calculate the maximum velocity, the velocity at the interface, the plane of zero shear stress, and the drag on the walls of the slit.

§2.6 CREEPING FLOW AROUND A SPHERE^{1,2,3,4}

In the preceding sections several elementary viscous flow problems have been solved. These have all dealt with rectilinear flows with only one nonvanishing velocity component. Since the flow around a sphere involves two nonvanishing velocity components, v_r and v_θ , it cannot be conveniently understood by the techniques explained at the beginning of this chapter. Nonetheless, a brief discussion of flow around a sphere is warranted here because of the importance of flow around submerged objects. In Chapter 4 we show how to obtain the velocity and pressure distributions. Here we only cite the results and show how they can be used to derive some important relations that we need in later discussions. The problem treated here, and also in Chapter 4, is concerned with “creeping flow”—that is, very slow flow. This type of flow is also referred to as “Stokes flow.”

We consider here the flow of an incompressible fluid about a solid sphere of radius R and diameter D as shown in Fig. 2.6-1. The fluid, with density ρ and viscosity μ , ap-

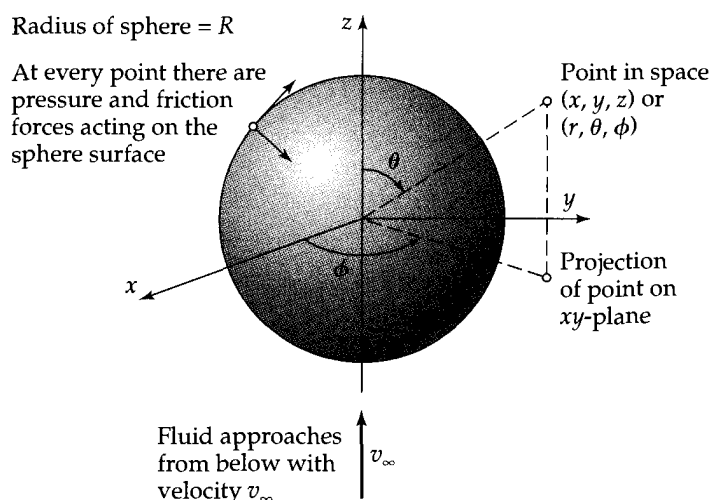


Fig. 2.6-1 Sphere of radius R around which a fluid is flowing. The coordinates r , θ , and ϕ are shown. For more information on spherical coordinates, see Fig. A.8-2.

¹ G. G. Stokes, *Trans. Cambridge Phil. Soc.*, **9**, 8–106 (1851). For creeping flow around an object of arbitrary shape, see H. Brenner, *Chem. Engr. Sci.*, **19**, 703–727 (1964).

² L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, 2nd edition, Pergamon, London (1987), §20.

³ G. K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge University Press (1967), §4.9.

⁴ S. Kim and S. J. Karrila, *Microhydrodynamics: Principles and Selected Applications*, Butterworth-Heinemann, Boston (1991), §4.2.3; this book contains a thorough discussion of “creeping flow” problems.

proaches the fixed sphere vertically upward in the z direction with a uniform velocity v_∞ . For this problem, "creeping flow" means that the Reynolds number $Re = Dv_\infty\rho/\mu$ is less than about 0.1. This flow regime is characterized by the absence of eddy formation downstream from the sphere.

The velocity and pressure distributions for this creeping flow are found in Chapter 4 to be

$$v_r = v_\infty \left[1 - \frac{3}{2} \left(\frac{R}{r} \right) + \frac{1}{2} \left(\frac{R}{r} \right)^3 \right] \cos \theta \quad (2.6-1)$$

$$v_\theta = v_\infty \left[-1 + \frac{3}{4} \left(\frac{R}{r} \right) + \frac{1}{4} \left(\frac{R}{r} \right)^3 \right] \sin \theta \quad (2.6-2)$$

$$v_\phi = 0 \quad (2.6-3)$$

$$p = p_0 - \rho g z - \frac{3}{2} \frac{\mu v_\infty}{R} \left(\frac{R}{r} \right)^2 \cos \theta \quad (2.6-4)$$

In the last equation the quantity p_0 is the pressure in the plane $z = 0$ far away from the sphere. The term $-\rho g z$ is the hydrostatic pressure resulting from the weight of the fluid, and the term containing v_∞ is the contribution of the fluid motion. Equations 2.6-1, 2, and 3 show that the fluid velocity is zero at the surface of the sphere. Furthermore, in the limit as $r \rightarrow \infty$, the fluid velocity is in the z direction with uniform magnitude v_∞ ; this follows from the fact that $v_z = v_r \cos \theta - v_\theta \sin \theta$, which can be derived by using Eq. A.6-33, and $v_x = v_y = 0$, which follows from Eqs. A.6-31 and 32.

The components of the stress tensor τ in spherical coordinates may be obtained from the velocity distribution above by using Table B.1. They are

$$\tau_{rr} = -2\tau_{\theta\theta} = -2\tau_{\phi\phi} = \frac{3\mu v_\infty}{R} \left[-\left(\frac{R}{r} \right)^2 + \left(\frac{R}{r} \right)^4 \right] \cos \theta \quad (2.6-5)$$

$$\tau_{r\theta} = \tau_{\theta r} = \frac{3}{2} \frac{\mu v_\infty}{R} \left(\frac{R}{r} \right)^4 \sin \theta \quad (2.6-6)$$

and all other components are zero. Note that the normal stresses for this flow are nonzero, except at $r = R$.

Let us now determine the force exerted by the flowing fluid on the sphere. Because of the symmetry around the z -axis, the resultant force will be in the z direction. Therefore the force can be obtained by integrating the z -components of the normal and tangential forces over the sphere surface.

Integration of the Normal Force

At each point on the surface of the sphere the fluid exerts a force per unit area $-(p + \tau_{rr})|_{r=R}$ on the solid, acting normal to the surface. Since the fluid is in the region of greater r and the sphere in the region of lesser r , we have to affix a minus sign in accordance with the sign convention established in §1.2. The z -component of the force

is $-(p + \tau_{rr})|_{r=R}(\cos \theta)$. We now multiply this by a differential element of surface $R^2 \sin \theta d\theta d\phi$ to get the force on the surface element (see Fig. A.8-2). Then we integrate over the surface of the sphere to get the resultant normal force in the z direction:

$$F^{(n)} = \int_0^{2\pi} \int_0^\pi (-(p + \tau_{rr})|_{r=R} \cos \theta) R^2 \sin \theta d\theta d\phi \quad (2.6-7)$$

According to Eq. 2.6-5, the normal stress τ_{rr} is zero⁵ at $r = R$ and can be omitted in the integral in Eq. 2.6-7. The pressure distribution at the surface of the sphere is, according to Eq. 2.6-4,

$$p|_{r=R} = p_0 - \rho g R \cos \theta - \frac{3}{2} \frac{\mu v_\infty}{R} \cos \theta \quad (2.6-8)$$

When this is substituted into Eq. 2.6-7 and the integration performed, the term containing p_0 gives zero, the term containing the gravitational acceleration g gives the buoyant force, and the term containing the approach velocity v_∞ gives the "form drag" as shown below:

$$F^{(n)} = \frac{4}{3} \pi R^3 \rho g + 2\pi \mu R v_\infty \quad (2.6-9)$$

The buoyant force is the mass of displaced fluid ($\frac{4}{3} \pi R^3 \rho$) times the gravitational acceleration (g).

Integration of the Tangential Force

At each point on the solid surface there is also a shear stress acting tangentially. The force per unit area exerted in the $-\theta$ direction by the fluid (region of greater r) on the solid (region of lesser r) is $+\tau_{r\theta}|_{r=R}$. The z -component of this force per unit area is $(\tau_{r\theta}|_{r=R}) \sin \theta$. We now multiply this by the surface element $R^2 \sin \theta d\theta d\phi$ and integrate over the entire spherical surface. This gives the resultant force in the z direction:

$$F^{(t)} = \int_0^{2\pi} \int_0^\pi (\tau_{r\theta}|_{r=R} \sin \theta) R^2 \sin \theta d\theta d\phi \quad (2.6-10)$$

The shear stress distribution on the sphere surface, from Eq. 2.6-6, is

$$\tau_{r\theta}|_{r=R} = \frac{3}{2} \frac{\mu v_\infty}{R} \sin \theta \quad (2.6-11)$$

Substitution of this expression into the integral in Eq. 2.6-10 gives the "friction drag"

$$F^{(t)} = 4\pi \mu R v_\infty \quad (2.6-12)$$

Hence the total force F of the fluid on the sphere is given by the sum of Eqs. 2.6-9 and 2.6-12:

$$F = \underbrace{\frac{4}{3} \pi R^3 \rho g}_{\text{buoyant force}} + \underbrace{2\pi \mu R v_\infty}_{\text{form drag}} + \underbrace{4\pi \mu R v_\infty}_{\text{friction drag}} \quad (2.6-13)$$

or

$$F = F_b + F_k = \underbrace{\frac{4}{3} \pi R^3 \rho g}_{\text{buoyant force}} + \underbrace{6\pi \mu R v_\infty}_{\text{kinetic force}} \quad (2.6-14)$$

⁵ In Example 3.1-1 we show that, for incompressible, Newtonian fluids, all three of the normal stresses are zero at fixed solid surfaces in all flows.

The first term is the *buoyant force*, which would be present in a fluid at rest; it is the mass of the displaced fluid multiplied by the gravitational acceleration. The second term, the *kinetic force*, results from the motion of the fluid. The relation

$$F_k = 6\pi\mu Rv_\infty \quad (2.6-15)$$

is known as *Stokes' law*.¹ It is used in describing the motion of colloidal particles under an electric field, in the theory of sedimentation, and in the study of the motion of aerosol particles. Stokes' law is useful only up to a Reynolds number $Re = Dv_\infty\rho/\mu$ of about 0.1. At $Re = 1$, Stokes' law predicts a force that is about 10% too low. The flow behavior for larger Reynolds numbers is discussed in Chapter 6.

This problem, which could not be solved by the shell balance method, emphasizes the need for a more general method for coping with flow problems in which the streamlines are not rectilinear. That is the subject of the following chapter.

EXAMPLE 2.6-1

Determination of Viscosity from the Terminal Velocity of a Falling Sphere

Derive a relation that enables one to get the viscosity of a fluid by measuring the terminal velocity v_t of a small sphere of radius R in the fluid.

SOLUTION

If a small sphere is allowed to fall from rest in a viscous fluid, it will accelerate until it reaches a constant velocity—the *terminal velocity*. When this steady-state condition has been reached the sum of all the forces acting on the sphere must be zero. The force of gravity on the solid acts in the direction of fall, and the buoyant and kinetic forces act in the opposite direction:

$$\frac{4}{3}\pi R^3\rho_s g = \frac{4}{3}\pi R^3\rho g + 6\pi\mu Rv_t \quad (2.6-16)$$

Here ρ_s and ρ are the densities of the solid sphere and the fluid. Solving this equation for the terminal velocity gives

$$\mu = \frac{2}{9}R^2(\rho_s - \rho)g/v_t \quad (2.6-17)$$

This result may be used only if the Reynolds number is less than about 0.1.

This experiment provides an apparently simple method for determining viscosity. However, it is difficult to keep a homogeneous sphere from rotating during its descent, and if it does rotate, then Eq. 2.6-17 cannot be used. Sometimes weighted spheres are used in order to preclude rotation; then the left side of Eq. 2.6-16 has to be replaced by m , the mass of the sphere, times the gravitational acceleration.

QUESTIONS FOR DISCUSSION

1. Summarize the procedure used in the solution of viscous flow problems by the shell balance method. What kinds of problems can and cannot be solved by this method? How is the definition of the first derivative used in the method?
2. Which of the flow systems in this chapter can be used as a viscometer? List the difficulties that might be encountered in each.
3. How are the Reynolds numbers defined for films, tubes, and spheres? What are the dimensions of Re ?
4. How can one modify the film thickness formula in §2.2 to describe a thin film falling down the interior wall of a cylinder? What restrictions might have to be placed on this modified formula?
5. How can the results in §2.3 be used to estimate the time required for a liquid to drain out of a vertical tube that is open at both ends?
6. Contrast the radial dependence of the shear stress for the laminar flow of a Newtonian liquid in a tube and in an annulus. In the latter, why does the function change sign?